Řetízkové pravidlo

Věta 20 (derivace složené funkce)**.** Nechť $r, s \in \mathbb{N}$ a nechť $G \subset \mathbb{R}^s, H \subset \mathbb{R}^r$ jsou otevřené množiny. Nechť $\varphi_1, \dots, \varphi_r \in C^1(G), f \in C^1(H)$ a bod $[\varphi_1(x), \dots, \varphi_r(x)] \in H$ pro každé $x \in G$. Potom složená funkce $F : G \to \mathbb{R}$ daná předpisem

$$F(x) = f\left(\varphi_1(x), \varphi_2(x), \dots, \varphi_r(x)\right), x \in G,$$

je třídy C^1 na G. Nechť $a \in G$ a $b = [\varphi_1(a), \dots, \varphi_r(a)]$. Pak pro $j \in \{1, \dots, s\}$ platí

$$\frac{\partial F}{\partial x_j}(a) = \sum_{i=1}^{r} \frac{\partial f}{\partial y_i}(b) \frac{\partial \varphi_i}{\partial x_j}(a).$$

Příklad 1. Vypočtěte všechny parciální derivace funkce f pomocí řetízkového pravidla.

(a) $f(u, v) = u\sqrt{1 + v^2}, u(x, y) = e^{2x}, v(x, y) = e^x$

(b) $f(u, v, w) = uv^2w^3, u(x, y) = -\sin x, v(x, y) = \cos x, v(x, y) = e^x$

(c) $f(u, v) = \sin u \cos v, u(x, y) = (x - y)^2, w(x, y) = x^2 - y^2$

(d) $f(u, v, w) = vw^2 - u^3, u(x, y, z) = e^{x-y}, v(x, y, z) = \log\left(x + 2y + 3z\right), w(x, y, z) = \sqrt{xy + z}$

Příklad 2. Ukažte, že funkce $F(x, y, z) = \frac{xy}{z} \log x + xf\left(\frac{y}{x}, \frac{z}{x}\right)$ vyhovuje vztahu $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = F + \frac{xy}{z}$.

Příklad 3. Nechť $g(x, y) = f(x + y, x - y)$. Spočtěte $\frac{\partial^2 g}{\partial x \partial y}$ v bodě $[a, b]$.

Příklad 4. Ukažte, že funkce $F(x, y) = xf(x+y)+yg(x+y)$ vyhovuje rovnici $\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2} = 0$.

Řešení

Příklad 1 (a)

$$\frac{\partial}{\partial x} f(u(x, y), v(x, y)) \overset{\text{V 20}}{=} f'_u(u(x, y), v(x, y)) \cdot u'_x(x, y) + f'_v(u(x, y), v(x, y)) \cdot v'_x(x, y) =$$

$$= \sqrt{1 + v(x, y)^2} \cdot 2e^{2x} + \frac{u(x, y)}{2\sqrt{1 + v(x, y)^2}} \cdot e^x = \sqrt{1 + (e^x)^2} \cdot 2e^{2x} + \frac{e^{2x}}{2\sqrt{1 + (e^x)^2}} \cdot e^x =$$

$$= 2e^{2x}\sqrt{1 + e^{2x}} + \frac{e^{3x}}{2\sqrt{1 + e^{2x}}}$$

$$\frac{\partial}{\partial y} f(u(x, y), v(x, y)) \overset{\text{V 20}}{=} f'_u(u(x, y), v(x, y)) \cdot u'_y(x, y) + f'_v(u(x, y), v(x, y)) \cdot v'_y(x, y) =$$

$$= \sqrt{1 + v(x, y)^2} \cdot 0 + \frac{u(x, y)}{2\sqrt{1 + v(x, y)^2}} \cdot 0 = 0$$

Příklad 1 (b)

$$\frac{\partial}{\partial x} f(u(x,y),v(x,y)) \overset{\text{V 20}}{=} f'_u \cdot u'_x + f'_v \cdot v'_x + f'_w \cdot w'_x =$$
$$= v(x,y)^2 w(x,y)^3 \cdot (-\cos x) + 2u(x,y)v(x,y)w(x,y)^3 \cdot (-\sin x) + 3u(x,y)v(x,y)^2 w(x,y)^2 \cdot e^x =$$
$$= -\cos^3 x \cdot e^{3x} + 2\sin^2 x \cos x e^{3x} - 3 \sin x \cos^2 x e^{3x}$$

$$\frac{\partial}{\partial y} f(u(x,y),v(x,y)) \overset{\text{V 20}}{=} f'_u \cdot u'_y + f'_v \cdot v'_y + f'_w \cdot w_y =$$
$$= v(x,y)^2 w(x,y)^3 \cdot 0 + 2u(x,y)v(x,y)w(x,y)^3 \cdot 0 + 3u(x,y)v(x,y)^2 w(x,y)^2 \cdot 0 = 0$$

Příklad 1 (c)

$$\frac{\partial}{\partial x} f\left(u(x,y),v(x,y)\right) \overset{\text{V 20}}{=} f'_u\left(u(x,y),v(x,y)\right) \cdot u'_x(x,y) + f'_v\left(u(x,y),v(x,y)\right) \cdot v'_x =$$
$$= \cos\left(u(x,y)\right)\cos\left(v(x,y)\right) \cdot 2(x-y) - \sin\left(u(x,y)\right)\sin v(x,y) \cdot 2x =$$
$$= 2(x-y)\cos\left((x-y)^2\right)\cos(x^2-y^2) - 2x \sin\left((x-y)^2\right)\sin\left(x^2-y^2\right)$$

$$\frac{\partial}{\partial y} f\left(u(x,y),v(x,y)\right) \overset{\text{V 20}}{=} f'_u\left(u(x,y),v(x,y)\right) \cdot u'_y(x,y) + f'_v\left(u(x,y),v(x,y)\right) \cdot v'_y =$$
$$= \cos\left(u(x,y)\right)\cos\left(v(x,y)\right) \cdot 2(x-y) \cdot (-1) - \sin\left(u(x,y)\right)\sin\left(v(x,y)\right) \cdot (-2y) =$$
$$= 2(y-x)\cos\left((x-y)^2\right)\cos(x^2-y^2) + 2y \sin\left((x-y)^2\right)\sin\left(x^2-y^2\right)$$

Příklad 1 (d)

$$\frac{\partial}{\partial x} f\left(u(x,y,z),v(x,y,z),w(x,y,z)\right) = f'_u \cdot u'_x + f'_v \cdot v'_x + f'_w \cdot w'_x =$$
$$= -3u(x,y)^2 \cdot e^x + w(x,y)^2 \cdot \frac{1}{x+2y+3z} + 2v(x,y)w(x,y) \cdot \frac{y}{2\sqrt{xy+z}} =$$
$$= -3e^{3x-2y} + \frac{xy+z}{x+2y+3z} + 2\log(x+2y+3z)\frac{y}{2}$$

$$\frac{\partial}{\partial y} f\left(u(x,y,z),v(x,y,z),w(x,y,z)\right) = f'_u \cdot u'_y + f'_v \cdot v'_y + f'_w \cdot w'_y =$$
$$= -3u(x,y)^2 \cdot \left(-e^{x-y}\right) + w(x,y)^2 \cdot \frac{2}{x+2y+3z} + 2v(x,y)w(x,y) \cdot \frac{x}{2\sqrt{xy+z}} =$$
$$= 3e^{3(x-y)} + \frac{2(xy+z)}{x+2y+3z} + x\log(x+2y+3z)$$

$$\frac{\partial}{\partial z} f\left(u(x,y,z),v(x,y,z),w(x,y,z)\right) = f'_u \cdot u'_z + f'_v \cdot v'_z + f'_w \cdot w'_z =$$
$$= -3u(x,y)^2 \cdot 0 + w(x,y)^2 \cdot \frac{3}{x+2y+3z} + 2v(x,y)w(x,y) \cdot \frac{1}{2\sqrt{xy+z}} =$$
$$= \frac{3\sqrt{xy+z}}{x+2y+3z} + \log(x+2y+3z)$$

Příklad 2
Pomocí řetízkového pravidla spočtěme parciální derivace funkce F.

$$\frac{\partial F}{\partial x} \overset{\text{V 20}}{=} \frac{y}{z}\log x + \frac{y}{z} + f\left(\frac{y}{x}, \frac{z}{x}\right) + x \cdot \left(f'_x\left(\frac{y}{x}, \frac{z}{x}\right) \cdot \frac{-y}{x^2} + f'_y\left(\frac{y}{x}, \frac{z}{x}\right) \cdot \frac{-z}{x^2}\right)$$
$$\frac{\partial F}{\partial y} \overset{\text{V 20}}{=} \frac{x}{z}\log x + x \cdot f'_x\left(\frac{y}{x}, \frac{z}{x}\right) \cdot \frac{1}{x}$$
$$\frac{\partial F}{\partial z} \overset{\text{V 20}}{=} \frac{-xy}{z^2}\log x + x \cdot f'_y\left(\frac{y}{x}, \frac{z}{x}\right) \cdot \frac{1}{x}$$

Pak platí následující.

$$\begin{aligned} x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} =& \frac{xy}{z}\log x + \frac{xy}{z} + xf\left(\frac{y}{x}, \frac{z}{x}\right) - yf'_x\left(\frac{y}{x}, \frac{z}{x}\right) - zf'_y\left(\frac{y}{x}, \frac{z}{x}\right) + \\ &+ \frac{xy}{z}\log x + yf'_x\left(\frac{y}{x}, \frac{z}{x}\right) - \frac{xy}{z}\log x + zf'_y\left(\frac{y}{x}, \frac{z}{x}\right) = \\ &= \frac{xy}{z}\log x + xf\left(\frac{y}{x}, \frac{z}{x}\right) + \frac{xy}{z} = F + \frac{xy}{z} \end{aligned}$$

Příklad 3

$$\frac{\partial g}{\partial x} \overset{\text{V 20}}{=} f'_x(x+y, x-y) + f'_y(x+y, x-y)$$

$$\begin{aligned} \frac{\partial^2 g}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(f'_x(x+y, x-y) + f'_y(x+y, x-y)\right) = \\ &\overset{\text{V 20}}{=} f''_{x,x}(x+y, x-y) + f''_{x,y}(x+y, x-y) \cdot (-1) + f''_{y,x}(x+y, x-y) + f''_{y,y}(x+y, x-y) \cdot (-1) \end{aligned}$$

$$\frac{\partial^2 g}{\partial x \partial x}(a, b) = f''_{x,x}(a+b, a-b) - f''_{x,y}(a+b, a-b) + f''_{y,x}(a+b, a-b) - f''_{y,y}(a+b, a-b)$$

Příklad 4 TODONějak to nevychází, celé přepočítat!
Spočtěme první a druhé parciální derivace F.

$$\frac{\partial F}{\partial x} \overset{\text{V 20}}{=} f(x+y) + xf'(x+y) + yg'(x+y)$$
$$\frac{\partial F}{\partial y} \overset{\text{V 20}}{=} xf'(x+y) + g(x+y) + yg'(x+y)$$
$$\frac{\partial^2 F}{\partial x^2} \overset{\text{V 20}}{=} \frac{\partial}{\partial x}\left(f(x+y) + xf'(x+y) + yg'(x+y)\right) = 2f'(x+y) + xf''(x+y) + yg''(x+y)$$
$$\frac{\partial^2 F}{\partial x \partial y} \overset{\text{V 20}}{=} \frac{\partial}{\partial y}\left(f(x+y) + xf'(x+y) + yg'(x+y)\right) = f'(x+y) + xf''(x+y) + g'(x+y) + yg''(x+y)$$
$$\frac{\partial^2 F}{\partial y^2} \overset{\text{V 20}}{=} \frac{\partial}{\partial y}\left(xf'(x+y) + g(x+y) + yg'(x+y)\right) = xf''(x+y) + g'(x+y) + g'(x+y) + yg''(x+y)$$

Pak platí.

$$\begin{aligned}\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2} &= 2f' + xf'' + yg'' - 2f' - 2xf'' - 2g' - 2yg'' + xf'' + g' + g' + yg'' = \\ &= 2f' - 2f' + xf'' + xf'' - 2xf'' + yg'' + yg'' - 2yg'' - 2g' + g' + g' = 0\end{aligned}$$